Another New Family of Binary Sequences with Six or eight-valued Correlations

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In this correspondence,for a positive even integer n, a new family of binary sequences with 2^n+1 sequences of length 2^n-1 taking six and eight valued correlations is presented. This family can be considered as a new class of Gold-like sequences.

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I. INTRODUCTION

Since the late sixties, many families of binary sequences of length 2^n-1 with optimal correlations [2],[3],[4],[6] have been found, where n is a positive integer. The Gold sequence family [2] is the best known binary sequence family having four-valued correlations. For an odd n, Boztas and Kumar [1] introduced a family of binary sequences , the so-called Gold-like sequences, whose correlation distribution is identical to that of Gold sequences. For even n, Udaya [7] introduced families of binary sequences with six-valued correlations. Later, Kim and No further generalized the Gold-like sequences to GKW-like sequences by the quadratic form technique [4]. In this paper, we use the quadratic form technique to get a new family of optimal binary sequences with 2^n+1 sequences of length 2^n-1.

II. PRELIMINARIES

Let F_{2^n} be the finite field with 2^n elements. The trace function from F_{2^n} to F_{2^e} is defined by

$$tr_e^n(x) = \sum_{i=0}^{\frac{n}{e}-1} x^{2^{ei}}$$

where $x \in F_{2^n}$ and $e|n$ and $\{v_0, v_1, ..., v_{2^n-1}\}$ is an enumeration of the elements in F_{2^n}. We also recall that the symplectic bilinear form of a trace form $f(x)$ is

$$B(x,z) = f(x) + f(z) + f(x+z) \text{ for } x, z \in F_{2^n}.$$

Let $f(x)$ be a function from F_{2^n} to F_2. The *trace transform* $F(\lambda)$ of $f(x)$ is defined by

$$F(\lambda) = \sum_{x \in F_{2^n}} (-1)^{f(x)+tr_1^n(x\lambda)}.$$

Lemma 1. *(Helleseth and Kumar [3]) Let $f(x)$ be a quadratic Boolean function on F_{2^n}. If the rank of $f(x)$ is $2h$, $2 \le 2h \le n$, then the distribution of the trace transform values is given by*

$$F(\lambda) = \begin{cases} 2^{n-h}, & 2^{2h-1}+2^{h-1} times \\ 0, & 2^n - 2^{2h} times \\ -2^{n-h}, & 2^{2h-1}-2^{h-1} times \end{cases}$$

where rank is the co-dimension of the radical of $f(x)$.

All the sequence families considered in this paper are constructed by using the trace function $a(x) = tr_1^n(x)$ and some quadratic form $b(x)$ as follows:

$$C = \{f_i(x) | 0 \le i \le 2^n, x \in F^*_{2^n}\}$$

where

$$f_i(x) = \begin{cases} a(v_i x) + b(x), & 0 \le i \le 2^{n-1} \\ a(x), & i = 2^n. \end{cases}$$

The correlation function between two sequences defined by $f_i(x)$ and $f_j(x)$ can be given by the function from F_{2^n} to the set of integers $\mathbb{Z}$ as

$$R_{i,j}(\delta) = \textstyle\sum_{x \in F^*_{2^n}} (-1)^{f_i(x)+f_j(\delta x)}$$

where $\delta \in F^*_{2^n}$ and it can be expressed as a trace transform

$$\begin{aligned} R_{i,j}(\delta) &= \sum_{x \in F^*_{2^n}} (-1)^{tr_1^n([v_i+v_j]x)+g(x)} \\ &= -1 + \sum_{x \in F_{2^n}} (-1)^{tr_1^n(x\lambda)+g(x)} \\ &= -1 + G(\lambda) \end{aligned}$$

where $g(x) = b(\delta x) + b(x)$ and $\lambda = v_i + v_j \in F_{2^n}$.

Definition 1. *Let $\frac{n}{e} = m$ be even. We define the Boolean functions $p(x)$ and $q(x)$ by*
$p(x) = \sum_{l=1}^{\frac{n}{2}-1} tr_1^n(x^{2^l+1})$, $q(x) = \sum_{l=1}^{\frac{m}{2}-1} tr_1^n(x^{2^{el}+1})$.

Definition 2. *(Udaya [7]) For an even integer $n = 2k \ge 4$, Udaya introduced the following family G*

$$g_i(x) = \begin{cases} tr_1^n(v_i x) + p(x) + tr_1^{\frac{n}{2}}(x^{2^{\frac{n}{2}}+1}), & 0 \le i \le 2^n - 1 \\ tr_1^n(x), & i = 2^n. \end{cases}$$

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Theorem 1. *(Udaya [7]) For the family G , the distribution of correlation values $R_{i,j}(\delta)$ are given as follows:*

$$\begin{cases} 2^n-1, & 2^n+1 \textit{ times} \\ -1, & 2^{2n-1}(2^{n-1}+2^{n-2})+2^{2n}-2 \textit{ times} \\ -1+2^k, & (2^{2n-1}-2)(2^{n-1}+2^{k-1}) \textit{ times} \\ -1-2^k, & (2^{2n-1}-2)(2^{n-1}-2^{k-1}) \textit{ times} \\ -1+2^{k+1}, & 2^{2n-1}(2^{n-3}+2^{k-2}) \textit{ times} \\ -1-2^{k+1}, & 2^{2n-1}(2^{n-3}-2^{k-2}) \textit{ times.} \end{cases}$$

Definition 3. *(Kim and No [4]) Let $\frac{n}{e}=m$ be an even integer, where $m \geq 4$. Kim and No introduced the following sequences S with six-valued correlations.*

$$s_i(x)=\begin{cases} tr_1^n(v_ix)+q(x)+tr_1^{\frac{n}{2}}(x^{2^{\frac{n}{2}}+1}), & 0 \leq i \leq 2^n-1 \\ tr_1^n(x), & i=2^n. \end{cases}$$

Theorem 2. *(Kim and No [4]) For the family S , the distribution of correlation values $R_{i,j}(\delta)$are given as follows:*

$$\begin{cases} 2^n-1, & 2^n+1 \textit{ times} \\ -1, & 2^{2n-e}(2^n-2^{n-2e})+(2^{2n}-2) \textit{ times} \\ -1+2^{\frac{n+2e}{2}}, & 2^{2n-e}(2^{n-2e-1}+2^{\frac{n-2e-2}{2}}) \textit{ times} \\ -1-2^{\frac{n+2e}{2}}, & 2^{2n-e}(2^{n-2e-1}-2^{\frac{n-2e-2}{2}}) \\ -1+2^{\frac{n}{2}}, & (2^{2n}-2^{2n-e}-2)(2^{n-1}+2^{\frac{n}{2}-1}) \textit{times} \\ -1-2^{\frac{n}{2}}, & (2^{2n}-2^{2n-e}-2)(2^{n-1}-2^{\frac{n}{2}-1}) \textit{times.} \end{cases}$$

In this paper we introduce a new family U which is a combination of G and S.

Definition 4. *Let $\frac{n}{e}=m \geq 4$ be even. We define the family U of binary sequences by*

$$u_i(x)=\begin{cases} tr_1^n(v_ix)+p(x)+q(x), & 0 \leq i \leq 2^n-1 \\ tr_1^n(x), & i=2^n. \end{cases}$$

For the correlation property of the family U, we have the following main result.

Theorem 3. *The distribution of correlation values of the family U is given as when e is odd*

Correlation $(R_{i,j}(\delta))$	*Number of times it appears*
2^n-1	2^n+1
-1	$2^{3n}+2^{2n}-2^{n+1}+2^e$ $+2^{n+2e-1}(2^{e-1}-2^{n-1}-1)-2$
$-1+2^{n-\frac{e-1}{2}}$	$(2^{e-2}+2^{\frac{e-3}{2}})(2^{n+e}-2)$
$-1-2^{n-\frac{e-1}{2}}$	$(2^{e-2}-2^{\frac{e-3}{2}})(2^{n+e}-2)$
$-1+2^{n-e+1}$	$(2^{2e-3}+2^{e-2})(2^{2n}-2^{n+e})$
$-1-2^{n-e+1}$	$(2^{2e-3}-2^{e-2})(2^{2n}-2^{n+e})$

and when e is even

Correlation $(R_{i,j}(\delta))$	*Number of times it appears*
2^n-1	2^n+1
-1	$2^{3n}+2^{2n}-2^{n+1}+2^{e+1}$ $+2^{n+e-2}(3\cdot 2^{e-1}-2^{n+2}-3)-2$
$-1+2^{n-\frac{e}{2}}$	$(2^{e-1}+2^{\frac{e-2}{2}})(2^{n+e-1}+2^n-2)$
$-1-2^{n-\frac{e}{2}}$	$(2^{e-1}-2^{\frac{e-2}{2}})(2^{n+e-1}+2^n-2)$
$-1+2^{n-\frac{e-2}{2}}$	$(2^{e-3}+2^{\frac{e-4}{2}})(2^{n+e-1}-2^n)$
$-1-2^{n-\frac{e-2}{2}}$	$(2^{e-3}-2^{\frac{e-4}{2}})(2^{n+e-1}-2^n)$
$-1+2^{n-e}$	$(2^{2e-1}+2^{e-1})(2^{2n}-2^{n+e})$
$-1-2^{n-e}$	$(2^{2e-1}-2^{e-1})(2^{2n}-2^{n+e})$

III. CORRELATION OF $p(x)+q(x)$

The following theorem describes the correlation of $p(x)+q(x)$.

Theorem 4. *The distribution of the trace transform values (cross-correlation values) of $p(x)+q(x)$ is given as*

$$\begin{cases} 2^{n-\frac{e-1}{2}}, & 2^{e-2}+2^{\frac{e-3}{2}} \textit{times} \\ 0, & 2^n-2^{e-1} \textit{times} \\ -2^{n-\frac{e-1}{2}}, & 2^{e-2}-2^{\frac{e-3}{2}} \textit{times} \end{cases} \quad \textit{when e is odd}$$

$$\begin{cases} 2^{n-\frac{e}{2}}, & 2^{e-1}+2^{\frac{e-2}{2}} \textit{times} \\ 0, & 2^n-2^e \textit{times} \\ -2^{n-\frac{e}{2}}, & 2^{e-1}-2^{\frac{e-2}{2}} \textit{times} \end{cases} \quad \textit{when e is even.}$$

Proof. For the proof we will be using some results from [6] and [7]. We have for $p'(x)=\sum_{l=1}^{\frac{n}{2}-1} tr_1^n(x^{2^l+1})+tr_1^{\frac{n}{2}}(x^{2^{\frac{n}{2}}+1})$, $B_{p'}(x,z)=tr_1^n(z(tr_1^n(x)+x))$ and for $q'(x)=\sum_{l=1}^{\frac{m}{2}-1} tr_1^n(x^{2^{el}+1})+tr_1^{\frac{n}{2}}(x^{2^{\frac{n}{2}}+1})$, $B_{q'}(x,z)$ $=tr_1^n(z(tr_e^n(x)+x))$. Then $B_{p+q}(x,z)=B_{p'+q'}(x,z)$ $=tr_1^n(z[tr_e^n(x)+tr_1^n(x)])$. So we need to find the number of $x \in F_{2^n}$ such that $tr_e^n(x)+tr_1^n(x)=0$ which implies $tr_e^n(x)=0$ or 1 when e is odd.
Now $tr_e^n : F_{2^n} \overset{onto}{\to} F_{2^e}$. So $|Ker(tr_e^n)|=\frac{2^n}{2^e}=2^{n-e}$. Therefore $|\{x \in F_{2^n} : tr_e^n(x)=0 \text{ or } 1\}|=2|Ker(tr_e^n)|=2^{n-e+1}$. Hence the rank of $p+q$ is $n-(n-e+1)=e-1$ and we get the first case using the Lemma 1.
When e is even, if $tr_e^n(x)=1, tr_1^n(x)=tr_1^e(tr_e^n(x))=tr_1^e(1)=0$. So $tr_e^n(x)+tr_1^n(x)\neq 0$. So when e is even $tr_e^n(x)=0$ and in that case rank of $p+q$ is equal to e and we get the second case. ☐

IV. PROOF OF THEOREM 8

The proof can be divided into the following five cases.
Case 1: $\delta=1, i=j$:
It is a trivial case and thus

$R_{i,j}(\delta) = \sum_{x\in F^*_{2^n}} (-1)^{f_i(x)+f_i(x)} = 2^n-1,\ 2^n+1$ times.

Case 2: $\delta \neq 1, i = j = 2^n$:

$R_{i,j}(\delta) = \sum_{x\in F^*_{2^n}} (-1)^{tr_1^n(x)+tr_1^n(\delta x)} = \sum_{x\in F^*_{2^n}} (-1)^{tr_1^n(1+\delta x)} = -1,\ 2^n-2$ times (number of choices for $\delta \neq 0, 1$).

Case 3: $\delta = 1, i \neq j, 0 \le i, j \le 2^n - 1$:

$$\begin{aligned} R_{i,j}(\delta) &= \sum_{x\in F^*_{2^n}} (-)^{u_i(x)+u_j(x)} \\ &= \sum_{x\in F^*_{2^n}} (-)^{tr_1^n((v_i+v_j)x)} \\ &= -1, \qquad 2^n(2^n-1) \text{ times} \end{aligned}$$

Case 4: $i = 2^n, j \neq 2^n$ (or $j = 2^n, i \neq 2^n$):
For fixed δ

$$\begin{aligned} R_{2^n,j}(\delta) &= \sum_{x\in F^*_{2^n}} (-)^{tr_1^n([\delta+v_j]x)+p(x)+q(x)} \\ &= -1 + \sum_{x\in F_{2^n}} (-)^{tr_1^n(\lambda x)+p(x)+q(x)} \end{aligned}$$

for $\lambda = \delta + v_j$.

The distribution of the trace transform of $p(x) + q(x)$ is already given in Theorem 9. Therefore, the distribution of correlation function for a fixed δ is given as

$$\begin{cases} -1+2^{n-\frac{e-1}{2}}, & 2^{e-2}+2^{\frac{e-3}{2}} \text{times} \\ -1, & 2^n-2^{e-1}\text{times} \\ -1-2^{n-\frac{e-1}{2}}, & 2^{e-2}-2^{\frac{e-3}{2}} \text{times} \end{cases} \quad \text{when } e \text{ is odd}$$

$$\begin{cases} -1+2^{n-\frac{e}{2}}, & 2^{e-1}+2^{\frac{e-2}{2}} \text{times} \\ -1, & 2^n-2^{e}\text{times} \\ -1-2^{n-\frac{e}{2}}, & 2^{e-1}-2^{\frac{e-2}{2}} \text{times} \end{cases} \quad \text{when } e \text{ is even.}$$

As δ varies over $F^*_{2^n}$, the distribution is for e odd

$$\begin{cases} -1+2^{n-\frac{e-1}{2}}, & (2^{e-2}+2^{\frac{e-3}{2}})(2^n-1) \text{ times} \\ -1, & (2^n-2^{e-1})(2^n-1) \text{ times} \\ -1-2^{n-\frac{e-1}{2}}, & (2^{e-2}-2^{\frac{e-3}{2}})(2^n-1) \text{ times} \end{cases}$$

and for e even

$$\begin{cases} -1+2^{n-\frac{e}{2}}, & (2^{e-1}+2^{\frac{e-2}{2}})(2^n-1) \text{ times} \\ -1, & (2^n-2^{e})(2^n-1) \text{ times} \\ -1-2^{n-\frac{e}{2}}, & (2^{e-1}-2^{\frac{e-2}{2}})(2^n-1) \text{ times} \end{cases}$$

Case 5: $\delta \in F_{2^n} \setminus \{0, 1\}$ and $0 \le i, j \le 2^n - 1$:
In this case, we have

$u_i(x)+u_j(\delta x) = p(x)+q(x)+p(\delta x)+q(\delta x)+tr_1^n([v_i+\delta v_j]x)$.

Actually, the correlation function is equivalent to the trace transform of a function $r(x)$ which is given as

$$r(x) = p(x) + q(x) + p(\delta x) + q(\delta x).$$

In order to compute the distribution of the correlation values, the rank of the symplectic form associated with $r(x)$ must be found and it is enough to count the number of x in F_{2^n} satisfying

$$B_r(x, z) = 0, \text{ for all } z \in F_{2^n}$$

where

$$B_r(x, z) = r(x) + r(z) + r(x + z).$$

Plugging $p(x)$ and $q(x)$ into $B_r(x, z)$, we have

$$\begin{aligned} B_r(x,z) &= tr_1^n(z[tr_1^n(x)+tr_e^n(x)] + \delta z[tr_1^n(\delta x)+tr_e^n(\delta x)]) \\ &= tr_1^n(z[\delta tr_1^n(\delta x)+\delta tr_e^n(\delta x)+tr_1^n(x)+tr_e^n(x)]). \end{aligned}$$

So the rank can be computed by determining the number of solutions to

$$\delta tr_1^n(\delta x) + \delta tr_e^n(\delta x) + tr_1^n(x) + tr_e^n(x) = 0. \quad (1)$$

Let $tr_e^n(x) = a$ and $tr_e^n(\delta x) = b$, where $a, b \in F_{2^e}$. Then (1) can be written as

$$\delta tr_1^e(b) + tr_1^e(a) + \delta b + a = 0. \quad (2)$$

SubCase 1: $\delta \in F_{2^e}$. Then from equation (1) , we get

$$\delta^2 a + \delta tr_1^n(\delta x) + tr_1^n(x) + a = 0 \quad (3)$$

as $tr_e^n(\delta x) = \delta tr_e^n(x) = \delta a$ because $\delta \in F_{2^e}$.
Now we have the following four cases depending on the values of $tr_1^n(\delta x)$ and $tr_1^n(x)$

1. $tr_1^n(\delta x) = 1$ and $tr_1^n(x) = 1$,which implies $\delta^2 a + \delta + 1 + a = 0$.
2. $tr_1^n(\delta x) = 1$ and $tr_1^n(x) = 0$, which gives $\delta^2 a+\delta+a = 0$.
3. $tr_1^n(\delta x) = 0$ and $tr_1^n(x) = 1$, which implies $\delta^2 a+1+a = 0$.
4. $tr_1^n(\delta x) = 0$ and $tr_1^n(x) = 0$, which gives $\delta^2 a + a = 0$.

The fourth equation is true only if $a = 0$, otherwise $(\delta^2+1)a = 0 \Rightarrow \delta = 1$ but $\delta \neq 0$ or 1. Also if $a = tr_e^n(x) = 0$, we get $tr_1^n(x) = tr_1^e(a) = tr_1^e(0) = 0$ and $tr_1^n(\delta x) = tr_1^e(\delta a) = tr_1^e(0) = 0$. So all the $x \in F_{2^n}$ for which $tr_e^n(x) = 0$ are solutions to the equation (3) and that actually gives us 2^{n-e} solutions so far. Before we start discussing the other two equations we need the following simple but interesting observation.

$$\begin{aligned} tr_1^e(\frac{\delta}{1+\delta^2}) &= tr_1^e(\frac{\delta+1}{\delta^2+1} + \frac{1}{\delta^2+1}) \\ &= tr_1^e(\frac{\delta+1}{\delta^2+1}) + tr_1^e(\frac{1}{\delta^2+1}) \\ &= tr_1^e(\frac{\delta+1}{(\delta+1)^2}) + tr_1^e((\frac{1}{\delta+1})^2) \\ &= tr_1^e(\frac{1}{\delta+1}) + tr_1^e(\frac{1}{\delta+1}) \\ &= 0. \end{aligned}$$

First, we consider e is odd. Then $tr_1^e(1) = 1$.
(1) gives $a = \frac{1}{1+\delta}$ which implies

$$\begin{aligned} tr_1^n(\delta x) &= tr_1^e(a\delta) \\ &= tr_1^e(\frac{\delta}{1+\delta}) \\ &= tr_1^e(1 + \frac{1}{\delta+1}) \\ &= tr_1^e(1) + tr_1^e(\frac{1}{\delta+1}) \\ &= 1 + tr_1^e(\frac{1}{\delta+1}) \end{aligned}$$

and $tr_1^n(x) = tr_1^e(a) = tr_1^e(\frac{1}{\delta+1})$.
(2) gives $a = \frac{\delta}{1+\delta^2}$ which implies

$$\begin{aligned} tr_1^n(\delta x) &= tr_1^e(\frac{\delta^2}{1+\delta^2}) \\ &= tr_1^e(\frac{\delta}{1+\delta}) \\ &= 1 + tr_1^e(\frac{1}{\delta+1}) \end{aligned}$$

and $tr_1^n(x) = tr_1^e(\frac{\delta}{1+\delta^2}) = 0$.
(3) gives $a = \frac{1}{1+\delta^2}$ which implies

$$tr_1^n(\delta x) = tr_1^e(\frac{\delta}{1+\delta^2}) = 0, tr_1^n(x) = tr_1^e(\frac{1}{1+\delta}).$$

Now if $tr_1^e(\frac{1}{1+\delta}) = 0$, then (2) works. If $tr_1^e(\frac{1}{1+\delta}) = 1$, then (3) works. So in any case we have all together $2 \cdot 2^{n-e} = 2^{n-e+1}$ solutions.

Now we consider the case when e is even. Then $tr_1^e(1) = 0$ and $tr_1^e(\frac{\delta}{1+\delta}) = tr_1^e(\frac{1}{1+\delta})$.
(1) gives $a = \frac{1}{1+\delta}$ which implies

$$\begin{aligned} tr_1^n(\delta x) &= tr_1^e(\frac{\delta}{1+\delta}) \\ &= tr_1^e(\frac{1}{\delta+1}) \\ &= tr_1^n(x). \end{aligned}$$

(2) gives $a = \frac{\delta}{1+\delta^2}$ which implies

$$\begin{aligned} tr_1^n(\delta x) &= tr_1^e(\frac{\delta^2}{1+\delta^2}) \\ &= tr_1^e(\frac{\delta}{1+\delta}) \\ &= tr_1^e(\frac{1}{\delta+1}) \end{aligned}$$

and $tr_1^n(x) = tr_1^e(\frac{\delta}{1+\delta^2}) = 0$.
(3) gives $a = \frac{1}{1+\delta^2}$ which implies just like before

$$tr_1^n(\delta x) = 0 \text{ and } tr_1^n(x) = tr_1^e(\frac{1}{1+\delta}).$$

Now if $tr_1^e(\frac{1}{1+\delta}) = 0$, then (1),(2) and (3) do not work, only (4) works and that gives us 2^{n-e} solutions. If $tr_1^e(\frac{1}{1+\delta}) = 1$, then each of (1),(2),(3) and (4) works and in that case we have $4 \cdot 2^{n-e} = 2^{n-e+2}$ many solutions. So actually, in half of the cases we have 2^{n-e} many solutions and the other half gives us 2^{n-e+2} many solutions.
As δ varies over $F_{2^e} \setminus \{0, 1\}$ and $0 \le i, j \le 2^n - 1$, the distribution of correlation function is

$$\begin{cases} -1 + 2^{n-\frac{e-1}{2}}, & (2^{e-2} + 2^{\frac{e-3}{2}})2^n(2^e - 2) \text{ times} \\ -1, & (2^n - 2^{e-1})2^n(2^e - 2) \text{ times} \\ -1 - 2^{n-\frac{e-1}{2}}, & (2^{e-2} - 2^{\frac{e-3}{2}})2^n(2^e - 2) \text{ times} \end{cases}$$

and when e is even

$$\begin{cases} -1 + 2^{n-\frac{e}{2}}, & (2^{e-1} + 2^{\frac{e-2}{2}})2^n(2^{e-1} - 1) \text{ times} \\ -1 + 2^{n-\frac{e-2}{2}}, & (2^{e-3} + 2^{\frac{e-4}{2}})2^n(2^{e-1} - 1) \text{ times} \\ -1, & (2^n - 2^e)2^n(2^{e-1} - 1) \\ , & +(2^n - 2^{e-2})2^n(2^{e-1} - 1) \text{ times} \\ -1 - 2^{n-\frac{e}{2}}, & (2^{e-1} - 2^{\frac{e-2}{2}})2^n(2^{e-1} - 1) \text{ times} \\ -1 - 2^{n-\frac{e-2}{2}}, & (2^{e-3} - 2^{\frac{e-4}{2}})2^n(2^{e-1} - 1) \text{ times} \end{cases}$$

SubCase 2: $\delta \notin F_{2^e}$. This case is little complicated. Say $n = se$ and pick $\delta \in F_{2^n}$. Consider the map $\phi : F_{2^n} \longrightarrow F_{2^e} \times F_{2^e}$ by $\phi(x) = (tr_e^n(x), tr_e^n(\delta x))$.
We claim: If $\delta \notin F_{2^e}$, then ϕ is onto.
proof: Set $q = 2^e$. Write $\delta = \epsilon^q$. Set $\delta' = \epsilon^q + \epsilon$. $\delta \notin F_q \Rightarrow \delta' \neq 0$, since $\delta' = 0 \Rightarrow \epsilon^q = \epsilon \Rightarrow \epsilon \in F_q \Rightarrow \delta \in F_q$. Pick any $z \in F_q$. tr_e^n is onto , so $\exists \gamma \in F_{q^s}$ with $tr_e^n(\gamma) = z$.
Set $\beta = (\delta')^{-1}\gamma$ (possible as $\delta' \neq 0$). Let $x = \beta^q + \beta$. Then $tr_e^n(x) = 0$ and

$$\begin{aligned} tr_e^n(\delta x) &= tr_e^n(\epsilon^q\beta^q + \epsilon^q\beta) \\ &= tr_e^n(\epsilon^q\beta^q + \epsilon\beta + \delta'\beta) \\ &= tr_e^n(\delta'\beta) \\ &= tr_e^n(\gamma) = z. \end{aligned}$$

So $(0, z) \in \mathrm{Im}(\phi)$ and since z is arbitrary $0 \times F_q \subset \mathrm{Im}(\phi)$. Similarly, $F_q \times 0 \subset \mathrm{Im}(\phi)$. As ϕ is additive, ϕ is onto.
Hence the number of solutions to $(tr_e^n(x), tr_e^n(\delta x)) = (\epsilon_1, \epsilon_2)$, where $\epsilon_i \in F_{2^e}$ is 2^{n-2e}. Now when $\delta \notin F_{2^e}$ from equation(2) we get $b = tr_1^e(b)$ and $a = tr_1^e(a)$.
Suppose, e is odd. Then $(b, a) = (1, 0), (0, 1), (1, 1)$ and $(0, 0)$ give us solutions to equation(2). So we have all together $4 \cdot 2^{n-2e} = 2^{n-2e+2}$ solutions. If e is even, none of $(b, a) = (1, 0), (0, 1), (1, 1)$ is a solution to equation(2). Consider $b = 1, a = 0$. Then $tr_1^e(b) = 0 \neq b$. Similarly, we can show that the other two do not work either. So only $(b, a) = (0, 0)$ gives us solution to equation (2) and we have 2^{n-2e} solutions. As δ varies over $F_{2^n} \setminus F_{2^e}$, the distribution of correlation function is

$$\begin{cases} -1 + 2^{n-e+1}, & (2^{2e-3} + 2^{e-2})2^n(2^n - 2^e) \text{ times} \\ -1, & (2^n - 2^{2e-2})2^n(2^n - 2^e) \text{ times} \\ -1 - 2^{n-e+1}, & (2^{2e-3} - 2^{e-2})2^n(2^n - 2^e) \text{ times} \end{cases}$$

$$\begin{cases} -1 + 2^{n-e}, & (2^{2e-1} + 2^{e-1})2^n(2^n - 2^e) \text{ times} \\ -1, & (2^n - 2^{2e})2^n(2^n - 2^e) \text{ times} \\ -1 - 2^{n-e}, & (2^{2e-1} - 2^{e-1})2^n(2^n - 2^e) \text{ times} \end{cases}$$

Combining the results of the above five cases, the distribution of the correlation values for the sequence family U can be obtained.

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